

Lecture 8 (Feb 22, 2016)

HW: 3.1 3.6 3.7 3.13 3.15

Chapter 4, Lyapunov Stability

4.13

4.1. Autonomous systems

stability of equilibrium points of nonlinear systems.

Methods to judge whether a system is stable, asymptotically stable, or unstable near an eq. pt. which

- 1) not restricted to hyperbolic eq. pt.
- 2) not restricted to planar systems.
- 3) useful for getting more than local results.

Given $\dot{x} = f(x)$, if $f(x^*) = 0$, then x^* is an eq. pt.

□ W.L.O.G we take $x^* = 0$:

suppose $x^* \neq 0$. Let $z = x - x^*$. Then

$$\dot{z} = \dot{x} = f(x) = f(z + x^*) =: g(z)$$

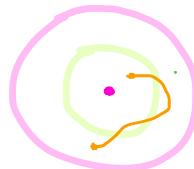
where $g(0) = f(x^*) = 0$.

□ Since system is autonomous, we can take to = 0.

Def x^* is

1) stable if $\forall \epsilon > 0$, $\exists \delta = \delta(\epsilon) > 0$ st.

$$\|x(0)\| < \delta \Rightarrow \|x(t)\| < \epsilon \quad \forall t > 0$$



2) unstable if it is not stable, i.e. if $\exists \epsilon > 0$ and $\exists T > 0$ and $x(0)$ where $\|x(0)\| < \delta$ s.t. $\|x(t)\| > \epsilon$ for $t = T$.

3) asymptotically stable if stable and $\lim_{t \rightarrow \infty} x(t) = 0$.

It is generally difficult to check for stability by this definition since we need to solve the equation.

\therefore we want methods that tell us stability without solving the ODE.

Example. Rigid space craft dynamics. $x = (x_1 \ x_2 \ x_3)^T$ is angular velocity in body fixed coordinates (aligned along principal axes).

Then, get Euler's equations:

$$\dot{x}_1 = \frac{I_2 - I_3}{I_1} x_2 x_3$$

$$\dot{x}_2 = \frac{I_3 - I_1}{I_2} x_1 x_3$$

$$\dot{x}_3 = \frac{I_1 - I_2}{I_3} x_1 x_2$$

$$\text{eq. pt: } x_1 = x_2 = x_3 = 0.$$

Let $E = \frac{1}{2} (I_1 x_1^2 + I_2 x_2^2 + I_3 x_3^2)$. Then

$$\begin{aligned}\dot{E} &= \frac{\partial E}{\partial x} \dot{x} = I_1 x_1 \left(\frac{I_2 - I_3}{I_1} \right) x_2 x_3 + \\ &\quad I_2 x_2 \left(\frac{I_3 - I_1}{I_2} \right) x_1 x_3 + \\ &\quad I_3 x_3 \left(\frac{I_1 - I_2}{I_3} \right) x_1 x_2 \\ &= x_1 x_2 x_3 (I_2 - I_3 + I_3 - I_1 + I_1 - I_2) = 0.\end{aligned}$$

so it says that dynamics start on energy ellipsoid $E(0) = c$, they stay on $E(t) = c$. So pick c so that $E=c$ is contained in $B_{\epsilon}(0)$. (Here can pick $\delta < \epsilon$ (or $\delta = \epsilon$) s.t. $E=c$ contains in $B_\delta(0) \subset B_\epsilon(0)$) so can conclude that $x=0$ is stable.

If there is damping in the dynamics: $\ddot{x} = f(x) - \begin{pmatrix} k_1 x_1 \\ k_2 x_2 \\ k_3 x_3 \end{pmatrix}$, then

$$\dot{E} = I_1 x_1 (-k_1 x_1) + I_2 x_2 (-k_2 x_2) + I_3 x_3 (-k_3 x_3) < 0 \quad \forall x \neq 0$$

So $E \rightarrow 0$ as $t \rightarrow \infty$ so $x \rightarrow 0$.

\therefore can conclude $x=0$ is asymptotically stable.

Lyapunov generalized notion of energy:

$V: D \rightarrow \mathbb{R}$ continuously differentiable function defined in a domain $D \subseteq \mathbb{R}^n$ that contains the origin $x^* = 0$.

Autonomous system: $\dot{x} = f(x)$, $f: D \rightarrow \mathbb{R}^n$ is locally Lip. on $D \subseteq \mathbb{R}^n$.

Theorem 4.1. Suppose

$V(0) = 0$ & $V(x) > 0 \quad \forall x \in D - \{0\}$ (positive definite V)

$\square \quad \dot{V} \leq 0 \text{ in } D \Rightarrow x=0 \text{ is stable}$
(negative semi-definite)

$\square \quad \dot{V} < 0 \text{ in } D - \{0\} \Rightarrow x=0 \text{ is asymptotically stable.}$
(negative definite)

V is called a Lyapunov function.

Remark 1. Don't need to solve ODE.

Remark 2. This gives sufficient condition only (converse thm, later)

Remark 3. In general, V is not constructive, although good first choice is to try $V = \frac{1}{2} x^T P x$, $P > 0$
(or V = energy of system)

Proof. Given $\epsilon > 0$, need to find $\delta > 0$ where $\delta < \epsilon$.

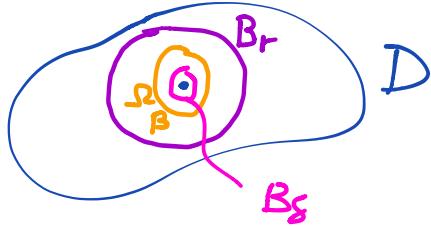
1) Find $0 < r < \epsilon$ so that $B_r(0) = \{x \in \mathbb{R}^n \mid \|x\| < r\} \subset D$.

2) Let $\alpha = \min_{\|x\|=r} V(x)$ (the minimum value of V on boundary of $B_r(0)$)

3) choose $\alpha < \beta < \alpha$. Let $\Omega_\beta = \{x \in B_r(0) \mid V(x) < \beta\}$

claim 1) Ω_β will be in the interior of $B_r(0)$.

2) Ω_β is a compact set.



Proof of claim 1) suppose not. Then, $\exists p \in \Omega_B$ s.t. $p \in \{x \in \mathbb{R}^n : \|x\| = r\}$.

$V(p) \geq \alpha > \beta$. However, $p \in \Omega_B \Rightarrow V(x) \leq \beta$.

\therefore claim is proved by contradiction. 2) closed & bounded.

4) $x(0) \in \Omega_B \rightarrow x(t) \in \Omega_B \quad \forall t \geq 0$. Because

$\dot{V} \leq 0$ in $D \Rightarrow V(x(t)) \leq V(x(0)) \leq \beta, \quad \forall t \geq 0$

Thus any trajectory starting in Ω_B (at $t=0$) stays in $\Omega_B, \forall t \geq 0$.

5) Ω_B : compact & invariant & f loc. Lip on D , by Theorem 3.3,
 $\dot{x} = f(x)$, $x(0) \in \Omega_B$ has a unique solution defined for all $t \geq 0$.

6) V : continuous, $V(0) = 0 \Rightarrow$ For $\beta > 0$, $\exists \delta > 0$ s.t.

$$\|x\| < \delta \rightarrow V(x) < \beta$$

Then, $B_\delta \subset \Omega_B \subset B_r \subset B_\varepsilon$ &

$$x(0) \in B_\delta \rightarrow x(0) \in \Omega_B \rightarrow x(t) \in \Omega_B \rightarrow x(t) \in B_r \rightarrow x(t) \in B_\varepsilon$$

$\Rightarrow \|x(0)\| < \delta \rightarrow \|x(t)\| < \varepsilon \quad \forall t \geq 0$. $\therefore x=0$ is stable.

Now suppose further that $\dot{V} < 0$ in $D - \{0\}$. Need to show that as

$t \rightarrow \infty$, $x(t) \rightarrow 0$, i.e., $\forall \alpha > 0$, $\exists T > 0$ s.t. $\|x(t)\| < \alpha \quad \forall t > T$.

As above given $\alpha > 0$, we can find $b > 0$ s.t. $\Omega_b \subset B_\alpha$. So if we can show that $V(x(t)) \rightarrow 0$ as $t \rightarrow \infty$, then we can conclude that $x(t) \rightarrow 0$ as $t \rightarrow \infty$ (since $V \rightarrow 0$ implies $\Omega_b \rightarrow 0$)

$$(\Omega_b = \{x \in B_\alpha : \|V(x)\| \leq b\}, \quad b < \min_{\|x\|=\alpha} V(x))$$

$V(x(t))$ monotonically decreasing & bounded from below ($V(x) \geq 0$)

$$\Rightarrow V(x(t)) \underset{t \rightarrow \infty}{\longrightarrow} c \geq 0.$$

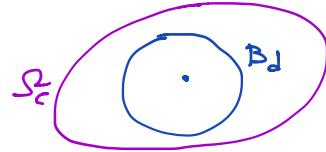
Claim. $c=0$.

Proof. suppose $c>0$. By continuity of V in x , given $c>0$, $\exists d>0$ s.t.

$$\|x\| < d \implies V(x) < c, \text{ i.e. } B_d \subset S_c$$

so if V stops at c , ($V(x) \geq c \implies x \notin B_d$)

then $x(t)$ lies outside B_d for all $t \geq 0$.



Let $-\gamma = \max_{d \leq \|x\| \leq r} \dot{V}(x)$ (exist since \dot{V} cont & $d \leq \|x\| \leq r$ compact)

Note $-\gamma < 0$. so

$$V(x(t)) = V(x(0)) + \int_0^t \dot{V}(x(\tau)) d\tau \leq V(x(0)) - \gamma t < 0 \quad t: \text{large}$$

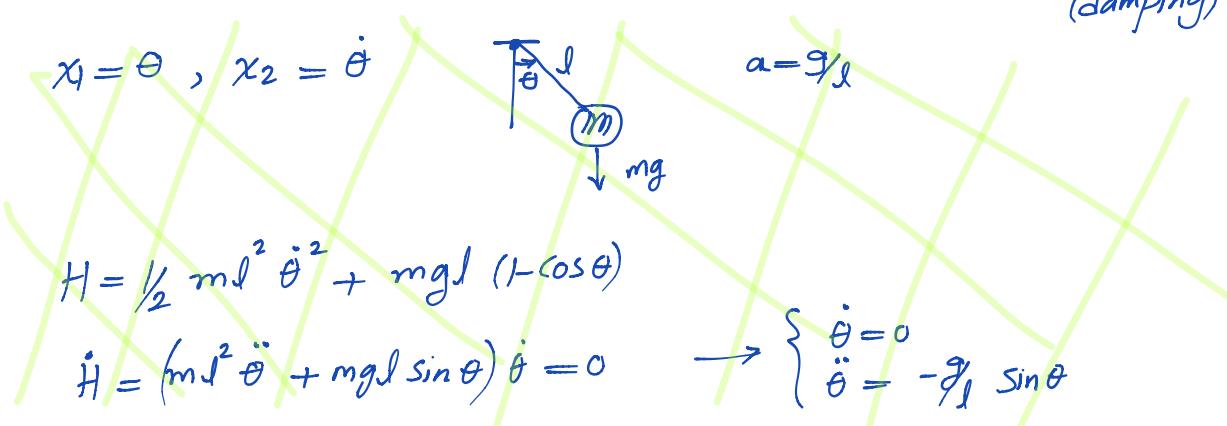
contradiction $\Rightarrow c=0$. ■

Def The set $\{x : V(x)=c\}$ is called a Lyapunov surface.

Example. Pendulum. $\dot{x}_1 = x_2$

$$\dot{x}_2 = -a \sin x_1 - b x_2$$

$a > 0, b > 0$
(damping)



$(0,0)$ & $(\pi,0)$ are two equilibrium points. We study $x=0$.

Let $V(x) = \frac{1}{2}x_2^2 + a(1-\cos x_1)$

$V(0) = 0$ & $V(x) > 0 \quad \forall x \neq 0$

$$\dot{V} = x_2 \dot{x}_2 + a \sin x_1 \dot{x}_1 = x_2(-a \sin x_1 - bx_2) + a \sin x_1 x_2 = -bx_2^2 \leq 0$$

\Rightarrow stability

But from this V cannot draw conclusion of asymptotical stability when $b > 0$, from Thm 4.1. Because $\dot{V} = 0$ when $x_2 = 0$ & x_1 arbitrary.

This example shows that 4.3 give a sufficient condition for stability & a.s. because we know that in pendulum, 0 is a.s.

Next we'll show how to handle this situation:

Idea: look at the set where $\dot{V} = 0 \rightarrow$ here is $x_2 = 0$

Dynamics on this set are $\begin{aligned} \dot{x}_1 &= 0 \\ 0 &= -a \sin x_1 \end{aligned}$ only solution is equil.

With the theory we have so far, want to choose a different V to prove a.s. for $b > 0$.

Consider $V(x) = \frac{1}{2}x^T Px + a(1-\cos x_1)$, $P > 0$

check that if choose $P = \begin{pmatrix} b/2 & b/2 \\ b/2 & 1 \end{pmatrix} > 0$, then

$$V = \frac{1}{4}(bx_1 + x_2)^2 + \frac{1}{4}x_2^2 + a(1-\cos x_1)$$

is positive def. & $V(0) = 0$ & $\dot{V} = -\frac{1}{2}bx_2^2 - \frac{1}{2}abx_1 \sin x_1 \leq 0$

for D defined st. $x_1 \in (-\pi, \pi)$. Then $\dot{V} < 0$ on $D - \{0\}$

$\Rightarrow x=0$ is a.s. by Thm 4.1.